

# Grassmannian fusion frames

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April 2, 2010

## Abstract

Transmitted data may be corrupted by both noise and data loss. Grassmannian frames are in some sense optimal representations of data transmitted over a noisy channel that may lose some of the transmitted coefficients. Fusion frame (or frame of subspaces) theory is a new area that has potential to be applied to problems in such fields as distributed sensing and parallel processing. Grassmannian fusion frames combine elements from both theories. A simple, novel construction of Grassmannian fusion frames shall be presented.

## 1 Introduction

### 1.1 Motivation

When data is transmitted over a communication line, the received message may be corrupted by noise and data loss. As an oversimplified example, if the message 1729 is sent to you, you could receive the noisy message 1728 or nothing at all. Representing data in a way that is resilient to such problems is clearly desirable. Expressing data using a redundant frame provides some protection, but some frames work better than others. Grassmannian frames are examples of such superior representations. Fusion frames are new objects which may be applied in a number of different fields and afford representations of higher dimensional measurements, as opposed to the one dimensional measurements of a traditional frame. We begin by laying a foundation in frame theory in Section 1.2. Then we specify in Section 2.1 and generalize in Section 2.2 by introducing Grassmannian frames and fusion frames. The main result, a construction of

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\*The author was supported in part by a Department of Education GAANN Fellowship, a University of Maryland Graduate School Ann G. Wylie Dissertation Fellowship, and a National Institutes of Health IRTA Postdoctoral Fellowship.

Grassmannian fusion frames utilizing Hadamard matrices, is in Section 3, followed by concluding remarks in Section 4.

## 1.2 Frames

Let  $\mathbb{F}$  be either  $\mathbb{R}$  or  $\mathbb{C}$ .

**Definition 1.** A sequence  $\{e_i\}_{i=1}^N$  in  $\mathbb{F}^M$  is a *frame* for  $\mathbb{F}^M$  if there exist constants  $0 < A \leq B < \infty$  such that

$$\forall x \in \mathbb{F}^M, \quad A\|x\|^2 \leq \sum_{i=1}^N |\langle x, e_i \rangle|^2 \leq B\|x\|^2. \quad (1)$$

$A$  and  $B$  are called *frame bounds*. A frame is *tight* if  $A = B$ . A frame is *unit-norm* if  $\|e_i\| = 1$  for  $1 \leq i \leq N$ . A frame is *equiangular* if for some  $\alpha$ ,  $|\langle e_i, e_j \rangle| = \alpha$  for all  $i \neq j$ .

Every orthonormal basis is a frame. One may view frames as generalizations of orthonormal bases which mimic the reconstruction properties (i.e.:  $\forall x, x = \sum \langle x, e_i \rangle e_i$ ) of orthonormal bases but may have some redundancy. We remark that  $\{e_i\}$  is a tight frame with frame bound  $A$  if and only if

$$\forall x \in \mathbb{F}^M, \quad AI_M x = \sum_{i=1}^N \langle x, e_i \rangle e_i, \quad (2)$$

where  $I_M$  is the identity matrix for  $\mathbb{F}^M$ .

## 2 Grassmannian fusion frames

### 2.1 Grassmannian frames

Goyal *et al.* proved that a unit-norm frame is *optimally robust against* (o.r.a.) noise and one erasure if the frame is tight [GKK01]. Furthermore, a unit-norm frame is o.r.a. multiple erasures if it is Grassmannian [SH03], [BK06].

**Definition 2.** Define

$$\mathcal{F}(N, \mathbb{F}^M) = \{\{e_i\}_{i=1}^N \subset \mathbb{F}^M : \{e_i\} \text{ is a unit-norm frame for } \mathbb{F}^M\}.$$

The *maximal frame correlation* is

$$\mathcal{M}_\infty(\{e_i\}_{i=1}^N) = \max_{1 \leq i < j \leq N} \{|\langle e_i, e_j \rangle|\}.$$

A sequence of unit-norm vectors  $\{u_i\}_{i=1}^N \subset \mathbb{F}^M$  is called a *Grassmannian frame* if it is a solution to

$$\min_{\{e_i\} \in \mathcal{F}(N, \mathbb{F}^M)} \{\mathcal{M}_\infty(\{e_i\}_{i=1}^N)\}.$$

If  $N = M$ , the Grassmannian frames are precisely the orthonormal bases for  $\mathbb{F}^M$ . If  $N = 3$  and  $M = 2$ , the 2-dimensional real vectors representing the cubic roots of unity are a Grassmannian frame. However, the vectors representing the fourth roots of unity do not form a Grassmannian frame for  $N = 4$  and  $M = 2$ . Since  $|\langle (1,0), (-1,0) \rangle| = 1$ , the fourth roots of unity actually have the largest possible maximal frame correlation. The following theorem is proven in a number of classical texts, see [SH03] for one proof and citations of other methods.

**Theorem 3.** *Let  $\{e_i\}_{i=1}^N$  be a unit-norm frame for  $\mathbb{F}^M$ . Then*

$$\mathcal{M}_\infty(\{e_i\}_{i=1}^N) \geq \sqrt{\frac{N-M}{M(N-1)}} \quad (3)$$

Equality holds in (3) if and only if  $\{e_i\}$  is an equiangular tight frame. Thus, equiangular unit-norm frames are automatically Grassmannian frames. However, equality can only hold for certain  $N$  and  $M$ . Bodmann and Paulsen proved a functorial equivalence between real equiangular frames and  $\alpha$ -regular 2-graphs, where  $\alpha$  depends on  $N$  and  $M$  [BP05]. An  $\alpha$ -regular 2-graph is a particular type of hypergraph. This correspondence can be used to characterize when equiangular frames exist. Other than the case  $N = M + 1$ , there are very few known pairs  $(N, M)$  which yield equiangular frames. Further, there are many pairs  $(N, M)$  for which it has been proven that no equiangular frames exist. When equiangular frames do not exist, it can be complicated to construct Grassmannian frames.

**Definition 4.** For  $1 \leq m \leq M$ , set  $G(M, m)$  to be the collection of  $m$  dimensional subspaces of  $\mathbb{F}^M$ .  $G(M, m)$  is called a *Grassmannian*.  $G(M, m)$  may be endowed with many mathematical structures, but we shall only be concerned with the metric space structure induced by the *chordal distance*

$$\text{dist}(\mathcal{W}_i, \mathcal{W}_j) = [m - \text{tr}(P_i P_j)]^{1/2},$$

for  $\mathcal{W}_i, \mathcal{W}_j \in G(M, m)$ , where  $P_i$  is the orthogonal projection onto  $\mathcal{W}_i$ .

The *Grassmannian packing problem* is the problem of finding  $N$  elements in  $G(M, m)$  so that the minimal distance between any two of them is as large as possible. Finding Grassmannian frames is equivalent to solving the Grassmannian packing problem for  $G(M, 1)$ . It is natural to ask what analytic structures one obtains by considering the Grassmannian packing problem for  $m > 1$ .

## 2.2 Fusion frames

*Fusion frames*, originally called *frames of subspaces*, were introduced by Casazza and Kutyniok in [CK04]. There are many potentially exciting applications of fusion frames in areas such as

coding theory [Bod07], distributed sensing [CKL08], and neurology [RGJ06].

**Definition 5.** A *fusion frame* for  $\mathbb{F}^M$  is a finite collection of subspaces  $\{\mathcal{W}_i\}_{i=1}^N$  in  $\mathbb{F}^M$  such that there exist  $0 < A \leq B < \infty$  satisfying

$$A\|x\|^2 \leq \sum_{i=1}^N \|P_i x\|^2 \leq B\|x\|^2,$$

where  $P_i$  is an orthogonal projection onto  $\mathcal{W}_i$ . If  $A = B$ , we say that the fusion frame is *tight*.

A tight fusion frame consisting of equal dimensional subspaces with equal pairwise chordal distances is an *equi-distance tight fusion frame*.

Similar to (2), a fusion frame is tight with bound  $A$  if and only if

$$\sum_{i=1}^N P_i = AI_M. \quad (4)$$

We note that a more general definition of fusion frame exists which incorporates weights [CCH<sup>+</sup>09]; however, that definition will not be used in this paper. Fusion frames may either be viewed as generalizations of frames or special types of frames. In the former sense, we are merely replacing the projections (modulo constant multiples) of vectors  $x \in \mathbb{F}^M$  (on line (1)) onto the subspace spanned by each frame vector with projections onto spaces of dimensions possibly higher than 1. In the latter sense, we may see a fusion frame as a frame with subcollections of frame vectors which group in *nice* ways. As is common in the literature, *nice* is an oversimplification of some very deep properties. In fact, splitting frames into such sub-collections is related to the (in)famous Feichtinger Conjecture [CKST08]. We shall make use of the following simple lemma in later proofs.

**Lemma 6.** Assume that  $\{\mathcal{W}_i\}_{i=1}^N$  is a collection of subspaces in  $\mathbb{F}^M$  with corresponding orthonormal bases  $\{e_j^i\}_{j=1}^{m_i}$ . Then  $\{\mathcal{W}_i\}_{i=1}^N$  is a tight fusion frame if

$$L^t = (e_1^1 | e_2^1 | \cdots | e_{m_1}^1 | e_1^2 | \cdots | e_{m_N}^N)$$

has equal-norm orthogonal rows.

*Proof.* It suffices to show that (4) holds. Label the rows of  $L^t$  as  $\{f_1, f_2, \dots, f_M\}$  and the rows of  $(e_1^i | e_2^i | \cdots | e_{m_i}^i)$  as  $\{f_1^i, f_2^i, \dots, f_{m_i}^i\}$ . Then

$$\begin{aligned} P_i &= (e_1^i | e_2^i | \cdots | e_{m_i}^i) (e_1^i | e_2^i | \cdots | e_{m_i}^i)^t \\ &= (f_1^i | f_2^i | \cdots | f_{m_i}^i)^t (f_1^i | f_2^i | \cdots | f_{m_i}^i) \\ &= (\langle f_k^i, f_\ell^i \rangle)_{k,\ell}, \end{aligned}$$

and

$$\sum_{i=1}^N P_i = \left( \sum_{i=1}^N \langle f_k^i, f_\ell^i \rangle \right)_{k,\ell} = (\langle f_k, f_\ell \rangle)_{k,\ell}.$$

□

A fusion frame is o.r.a. against noise if it is tight [KPCL09], a tight fusion frame is o.r.a. one subspace erasure if the dimensions of the subspaces are equal [Bod07], and a tight fusion frame is o.r.a. multiple subspace erasures if the subspaces have equal chordal distances. An equi-distance tight fusion frame is a solution to the Grassmannian packing problem [KPCL09]. The following definition does not exist in the literature but seems quite natural.

**Definition 7.** A fusion frame  $\{\mathcal{W}_i\}_{i=1}^N$  for  $\mathbb{F}^M$  consisting of  $d$ -dimensional subspaces shall be called a *Grassmannian fusion frame* if it is a solution to the Grassmannian packing problem for  $N$  points in  $G(M, m)$ .

We would like to construct such objects. The idea is very simple and makes use of Hadamard matrices.

## 3 Hadamard construction

### 3.1 Hadamard matrices

The first Hadamard matrices were discovered by Sylvester in 1867 [Syl67]. In 1893, Hadamard first defined and started to characterize Hadamard matrices, which have the maximal possible determinant amongst matrices with entries from  $\{\pm 1\}$  [Had93].

**Definition 8.** A *Hadamard Matrix* of order  $n$  is an  $n \times n$  matrix  $H$  with entries from  $\{\pm 1\}$  such that  $HH^t = nI_n$ .

In the original paper by Hadamard, it was proven that Hadamard matrices must have order equal to 2 or a multiple of 4. It is still an open conjecture as to whether Hadamard matrices exist for every dimension divisible by 4. However, there are many constructions of Hadamard matrices, which use methods from number theory, group cohomology and other areas of math. Horadam's book [Hor07] is an excellent resource. One class of Hadamard matrices are formed from Walsh functions.

**Definition 9.** The *Walsh functions*  $\omega_j : [0, 1] \rightarrow \{\pm 1\} \subset \mathbb{R}$ ,  $j \geq 0$ , are piecewise constant functions which have the following properties:

1.  $\omega_j(0) = 1$  for all  $j \geq 0$ ,

2.  $\omega_j$  has precisely  $j$  sign changes (zero crossings), and
3.  $\langle \omega_j, \omega_k \rangle = \delta_{jk}$ .

Walsh functions have been used for over 100 years by communications engineers to minimize cross talk. The first four Walsh functions are

$$\begin{aligned}\omega_0 &= \mathbb{1}_{[0,1]}, \\ \omega_1 &= \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1]}, \\ \omega_2 &= \mathbb{1}_{[0,1/4)} - \mathbb{1}_{[1/4,3/4)} + \mathbb{1}_{[3/4,1]}, \text{ and} \\ \omega_3 &= \mathbb{1}_{[0,1/4)} - \mathbb{1}_{[1/4,1/2)} + \mathbb{1}_{[1/2,3/4)} - \mathbb{1}_{[3/4,1]},\end{aligned}$$

where  $\mathbb{1}_A$  is the function that takes the value 1 on  $A$  and 0 off of  $A$ ,  $A$  measurable. By sampling the first  $2^n$  Walsh functions at the points  $\frac{k}{2^n}$ , one obtains a Hadamard matrix; that is,

$$W_n = \left( \omega_j \left( \frac{k}{2^n} \right) \right)_{0 \leq j, k < 2^n}$$

is a  $2^n \times 2^n$  Hadamard matrix. There is a speedy algorithm, the Fast Hadamard (or Walsh) Transform, for multiplying a vector by such a matrix.

### 3.2 Construction

The idea behind the new construction of the Grassmannian fusion frames is very simple. We remove the first  $j$  rows of a Hadamard matrix  $H$  to obtain a submatrix  $H'$ . The columns of  $H'$  will then be partitioned into spanning sets for subspaces. Since the elements of the matrix are  $\pm 1$ , the computations should be simplified and the resulting (fusion) frame should be easy to implement. This construction can be seen as similar to Naimark Complementation [CCH<sup>+</sup>09], although we start out knowing the larger Hilbert space. It is well-known (see, for example, [Hor07]) that any Hadamard matrix can be normalized so that the first row consists solely of 1s.

**Theorem 10.** *Let  $H$  be an  $n \times n$  Hadamard matrix indexed by  $0, \dots, n-1$  which has been normalized so that the first row consists solely of 1s. Then*

$$\{e_i = \frac{1}{\sqrt{n-1}}(H(i, k))_{1 \leq k \leq n-1} : 0 \leq i \leq n-1\}$$

*is a unit-norm Grassmannian frame for  $\mathbb{F}^{n-1}$  with frame bound  $\frac{n}{n-1}$ .*

*Proof.* For any  $0 \leq i_1, i_2 \leq n-1$ ,

$$\begin{aligned}\langle H(j, i_1)_{0 \leq j \leq n-1}, H(j, i_2)_{0 \leq j \leq n-1} \rangle &= n\delta_{i_1, i_2} \text{ and} \\ \langle H(i_1, j)_{0 \leq j \leq n-1}, H(i_2, j)_{0 \leq j \leq n-1} \rangle &= n\delta_{i_1, i_2}.\end{aligned}$$

Thus  $\langle e_{i_1}, e_{i_2} \rangle = \frac{1}{n-1}(n\delta_{i_1, i_2} - 1)$ , and the collection is equiangular. We now show that the  $e_i$  do indeed form a frame. Let  $x \in \mathbb{F}^{n-1}$  be arbitrary. We verify that (2) holds. Let

$$L = \left( \frac{1}{\sqrt{n-1}} (H(j, i)) \right)_{0 \leq i \leq n-1, 1 \leq j \leq n-1}.$$

Then, by the orthogonality of the columns,

$$\sum_{i=0}^{n-1} \langle x, e_i \rangle e_i = L^t L x = \frac{n}{n-1} x.$$

□

We now present the main construction.

**Theorem 11.** *Let  $W_n$  be the  $2^n \times 2^n$  Walsh-Hadamard matrix indexed by  $0, \dots, 2^n - 1$ . Then*

$$\{\mathcal{W}_i = \text{span}_{0 \leq k \leq 2^m-1} \{(W_n(j, i + k2^{n-m}))_{2^m \leq j \leq 2^n-1}\}\}_{0 \leq i \leq 2^{n-m}-1}$$

*is a tight Grassmannian fusion frame for  $\mathbb{F}^{2^n-2^m}$  with frame bound  $\frac{2^n}{2^n-2^m}$  consisting of  $2^{n-m}$   $2^m$ -dimensional subspaces.*

*Proof.* Note that the first  $2^m$  Walsh functions are constants over intervals of the form  $[\frac{\ell}{2^m}, \frac{\ell+1}{2^m})$ ,  $\ell \in \mathbb{Z}$ . Since the  $2^n \times 2^n$  Walsh-Hadamard matrix is generated by sampling at points of the form  $\frac{\ell}{2^n}$ , the submatrix of  $W_n$  formed from the first  $2^m$  rows consists of the first column of  $W_m$ , repeated  $2^{n-m}$  times, followed by the second column of  $W_m$ , repeated  $2^{n-m}$  times, and so on. Thus, for  $0 \leq i_1, i_2 \leq 2^{n-m} - 1$ ,

$$\begin{aligned} & \langle (W_n(j, i_1 + k_1 2^{n-m}))_{0 \leq j \leq 2^m-1}, (W_n(j, i_2 + k_2 2^{n-m}))_{0 \leq j \leq 2^m-1} \rangle = 2^m \delta_{k_1, k_2} \\ \Rightarrow & \langle (W_n(j, i_1 + k_1 2^{n-m}))_{2^m \leq j \leq 2^n-1}, (W_n(j, i_2 + k_2 2^{n-m}))_{2^m \leq j \leq 2^n-1} \rangle = 2^n \delta_{i_1, i_2} \delta_{k_1, k_2} - 2^m \delta_{k_1, k_2}. \end{aligned}$$

Set  $e_k^i = \frac{1}{\sqrt{2^n-2^m}} (W_n(j, i + k2^{n-m}))_{2^m \leq j \leq 2^n-1}$ . For each  $0 \leq i \leq 2^{n-m} - 1$ ,  $\{e_k^i\}_{0 \leq k \leq 2^m-1}$  is an orthonormal basis for  $\mathcal{W}_i$ . It follows from Lemma 6 that  $\{\mathcal{W}_i\}_{i=0}^{2^{n-m}-1}$  is a tight fusion frame with frame bound  $\frac{2^n}{2^n-2^m}$ . Further, for  $i_1 \neq i_2$ ,

$$\begin{aligned} \text{trace } P_{i_1} P_{i_2} &= \text{trace} (e_1^{i_1} | e_2^{i_1} | \dots | e_{m_i}^{i_1}) (e_1^{i_1} | e_2^{i_1} | \dots | e_{m_i}^{i_1})^t (e_1^{i_2} | e_2^{i_2} | \dots | e_{m_i}^{i_2}) (e_1^{i_2} | e_2^{i_2} | \dots | e_{m_i}^{i_2})^t \\ &= \text{trace} (e_1^{i_1} | e_2^{i_1} | \dots | e_{m_i}^{i_1}) \left( \frac{-2^m}{2^n - 2^m} I_{2^m} \right) (e_1^{i_2} | e_2^{i_2} | \dots | e_{m_i}^{i_2})^t \\ &= \frac{-2^m}{2^n - 2^m} \sum_{k=0}^{2^m} \langle e_k^{i_1}, e_k^{i_2} \rangle \\ &= \frac{-2^m}{2^n - 2^m} \left( 2^m \frac{-2^m}{2^n - 2^m} \right) \\ &= \frac{2^m}{(2^{n-m} - 1)^2}. \end{aligned}$$

Thus between any two distinct  $\mathcal{W}_i$ , the chordal distance is  $(2^m - \frac{2^m}{(2^{n-m}-1)^2})^{1/2}$  and the  $\mathcal{W}_i$  form a Grassmannian fusion frame. □

## 4 Conclusion

By utilizing well-known functions, we were able to construct equi-distance tight fusion frames in a very straightforward manner. The construction also automatically yields orthonormal bases for fusion frame subspaces which have very simple coordinates, namely  $\{\pm 1\}$ , which are easier to implement. The theory of Grassmannian fusion frames has exciting potential. We hope that the theory will expand beyond the equi-distance constructions presented in this paper and in [KPCL09].

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